Probability Theory
2024/25 Period IIb
Instructor: Gilles Bonnet
Exam
20/6/2024
Duration: 2 hours

Name: ______________________________

Student number: ______________________________

This exam contains 5 problems. Enter all requested information on the top of this page.

Your answers should be written in this booklet. Avoid handing in extra paper. In case you hand in extra paper and/or write parts of your answers in non obvious places, mark this explicitly so that it is not missed out during the grading process.

Do not use red ink.

You are **not** allowed to use any electronic devices during the exam.

You are **not** allowed to use any books, lecture notes or handwritten notes.

Each answer must be **justified**, unless explicitly stated otherwise.

Do not write on the table below.

Problem	Points	Score
1	15	
2	15	
3	20	
4	15	
5	25	
Total:	90	

Satisfying the above instructions gives you 10 extra free points.

1. A traditional fair die is rolled 4 times. Give the probability of the following events.
   Remark 1: The justification of your answers can be limited to a brief computation (no need to write many words).
   Remark 2: Give your answer in the form of a fraction of integer numbers.
   (a) (5 points) A six turns up exactly twice.
   (b) (5 points) All numbers are odd.
   (c) (5 points) The sum of the four rolls is 6.

2. (15 points) Let $X_1, X_2, \ldots, X_n$ be independent exponential random variables with expectation $\lambda > 0$. Show that the sum $S_n = X_1 + X_2 + \ldots + X_n$ has a gamma distribution with parameters n and λ^{-1}.
   Remark (to avoid confusion with the parametrization): the gamma distribution with parameters n and λ^{-1} has pdf proportional to $x^{n-1}e^{-x/\lambda}$ for $x > 0$.
   Hint: You might want to do an induction on n.

3. Let (X, Y) be a continuous random vector with joint density function given (up to a constant) by:

$$f_{X,Y}(x, y) = c(x + y)e^{-(x+y)}, \quad \text{for } x > 0,\ y > 0,$$

and 0 elsewhere.

(a) (15 points) Compute the marginal densities $f_X(x)$ and $f_Y(y)$ and the constant c.
Hint: You might want to start by computing the marginals up to the constant c and then use this to compute c.

(b) (5 points) Are X and Y independent? Justify your answer.

4. Consider an experiment where a fair coin is tossed $n = 10\,000$ times. Let X denote the number of heads observed.

   (a) (5 points) Give the expectation and variance of X.
   Remark: You do not need to justify your answer.

   (b) (10 points) Show that $\mathbb{P}(|X - \mathbb{E}[X]| \geq 500) \leq 0.01$.

5. A deck contains exactly 10 cards, of which 4 are red and 6 are black. The cards are shuffled and drawn one by one, without replacement. Let n be the number of cards drawn, with $n \leq 10$. Define the random variables $X_1, \ldots, X_n$ such that $X_i = 1$ if the i-th card drawn is red, and 0 otherwise. Let $S_n = X_1 + \cdots + X_n$.

   (a) (2 points) What is the variance of S_{10}?
   Remark: You don't need to justify your answer.

   (b) (5 points) What are the expectation and variance of X_i?
   Remark: You do not need to justify your answer.

   (c) (5 points) Compute $\mathbb{E}[X_i X_j]$ for $i \neq j$.

   (d) (5 points) Compute $\mathrm{Cov}(X_i, X_j)$ for $i \neq j$.

   (e) (8 points) Compute the variance of S_n.
   Sanity check: You might want to check compare with your answer of the first question of this exercise.